

Seeing relativity – I. Basics of a raytracing code in a Schwarzschild metric

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We present here an implementation of a raytracing code in the Schwarzschild metric. We aim at building a numerical code with a correct implementation of both special (aberration, amplification, Doppler) and general (deflection of light, lensing, gravitational redshift) relativistic effect by paying attention to a good rendering of stars.

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I. INTRODUCTION

The visual aspect of black holes is a frequent public outreach question but for decades, did not generate much interest among physicists or astronomers. A somewhat caricatural example of this is given by S. Chandrasekhar famous book on black holes [1], where the visual aspect of a Schwarzschild black hole is summarized in a tiny picture on page 130, the caption of which being rather pedantic and obscure for a non expert reader since the author does not describe the picture as a sketch of the angular size of a Schwarzschild black hole as a function of the distance of a static observer, but rather talks about the “cone of avoidance” of null geodesics (which technically means the same thing).

Still, the problem of black hole visualization had already drawn some sparse attention at the time of Chandrasekhar book in the more difficult context of the Kerr metric, the earliest work being that of Bardeen in 1972 [2] and later Luminet [3]. An increasing number of work were published afterward, a very incomplete subset of which lies in Refs. [4–10].

In the recent years, a much larger amount of work has been performed on black hole visualization. The main reason for this growing interest came from obvious astronomical constraints: the largest black hole (in term of angular size) seen from Earth, Sgr A*, has an angular diameter of order of $60 \mu\text{as}$, which is unobservable by conventional astronomical devices, but which should be at reach within less than a decade with the advent of long baseline interferometry in the millimetric domain, thus making the actual aspect of a black hole silhouette become a problem of astronomical relevance. Consequently, most of those recent works are motivated by actual astrophysical observational projects of our Galactic center such as GRAVITY [11] or the Event Horizon Telescope [12], see, e.g., [13–16], but some others were focused on what could be seen if an observer stood close to a black hole [17, 18] thus being less relevant from an observational perspective, but more focused on the diversity of physical effects that can arise in the vicinity of a

black hole. Our work fits in this second category.

The case of special relativity is much simpler and has deserved an earlier attention as early as 70 years ago with the pioneering sketches of Gamow [19]. (Although it is often said that special relativity was also a source of artistic inspiration for S. Dali in his famous painting “La persistència de la Memòria” (The Persistence of Memory, 1931), it actually does not seem to be the case [20].) The increasingly easier access to large computing facilities has progressively allowed the completion of excellent works by, for example Ruder and Nollert [21] or Searle *et al.* [22].

The aim of this paper is to present here a numerical code that implements most of the relativistic effects that arise when simulating what an observer would see in a black hole metric, focusing here on the Schwarzschild one. The Schwarzschild metric is the simplest black hole metric that exists. It is both astrophysically relevant (contrarily to the Reissner-Nordstrom one) and simple to study thanks to its spherical symmetry (contrarily to the Kerr or Kerr-Newmann metrics). In particular, as we shall see, it is possible to perform a rather nice and efficient rendering of pointlike light sources (i.e., stars) thanks to spherical symmetry, an issue which was not addressed satisfactorily till now.

The paper is structured as follows. The camera (i.e. the way to project a part of the celestial sphere on a two-dimensional screen) is described in Sec. II. We then address the way to simulate the way a background sky is distorted both by special relativistic effects and the presence of a gravitational field (Sec. III). This first, naive method can be significantly improved in term of computational time as explained in Sec. IV. The results of this section are then used to make a very rapid and satisfactory rendering of the stars (or any pointlike light source), as explained in Sec. V. In Sec. VI, we list all the data that we use in order to produce images, some examples of which are shown in Sec. VII. We then conclude in Sec. VIII.

In what follows, we shall use the $(+---)$ convention for the metric signature. We place ourselves in a coordinates systems such that $c = G = 1$, and keep the second as time unit, so that a distance of 1 corresponds to one light second (i.e. approximatively 3×10^5 km) and a mass

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of 1 corresponds to around 2×10^5 Solar masses. Unless otherwise specified we shall place ourselves in a spherical version of the so-called Schwarzschild coordinates, where the line element is written

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (1)$$

II. DEFINING THE OBSERVER AND ITS CAMERA

We assume that the images we want to simulate are those that would be seen by an observer is endowed with a four velocity u_{obs}^μ that either result from a chosen trajectory (possibly following a geodesic or not) or from any user defined data. We do not take into account the fact that the observer is unlikely to survive to his/her journey within the black hole metric either because of arbitrarily large acceleration imposed by some non geodesic motion or because of arbitrarily large tidal effects endured close or within a black hole of sufficiently small mass. We simulate here standard image and do not address the interesting issue of stereoscopic vision as was done in Ref. [23].

The camera orientation is described by a set of three unit orthogonal spacelike vectors, X^μ , Y^μ , Z^μ all of which are orthogonal to u_{obs}^μ . We define the orientation of the camera by the following assumptions:

- Any pixel of the screen can be seen as pointing toward a spacelike direction N^μ belonging to the X^μ , Y^μ , Z^μ hyperplane, where $N_\mu N^\mu = -1$, i.e., a photon hitting the screen coming from this direction possesses a wave vector proportional to $u_{\text{obs}}^\mu - N^\mu$, which is evidently a null vector;
- The Z^μ direction points toward the center of the screen, i.e. photons hitting the center of the screen possesses a wavevector proportional to $u_{\text{obs}}^\mu - Z^\mu$;
- The X^μ direction is associated to the central horizontal line of the screen in the sense that any photon hitting this part of the screen possesses a wavevector proportional to $u_{\text{obs}}^\mu - aZ^\mu - cX^\mu$, where $a^2 + c^2 = 1$, the value of a and c being determined by the pixel position and the choice of projection (see below);
- The Y^μ direction is associated to the central vertical line of the screen in the sense that any photon hitting this part of the screen possesses a wavevector proportional to $u_{\text{obs}}^\mu - a'Z^\mu - bY^\mu$, where $a'^2 + b^2 = 1$ (same remark as for a and c above).

The exact association between a pixel of coordinates (i, j) and the corresponding direction $N^\mu(i, j)$ is somewhat arbitrary and depends on the choice of projection we want to use. A natural choice is spherical projection, which reproduces the exact view on the screen provided

that the person viewing the screen is set at the proper position with respect to it, that depends on the view opening angle. When not at the proper position, any shape on the screen becomes distorted. For example, the circular silhouette of a Schwarzschild black hole no longer appears circular. In order to evade this problem, a natural choice corresponds to stereographic projection for which the circular shape of the black hole silhouette is always a circle on the screen. If one denotes α the opening angle of the picture along the horizontal direction, and R and C the number of rows and columns of the screen then the pixel coordinates (i, j) of a given direction N^μ is given by

$$i = \frac{C}{2} + 0.5 - \frac{C}{\cos \alpha/2} X_\mu \frac{N^\mu + Z^\mu}{1 - N_\mu Z^\mu}, \quad (2)$$

$$j = \frac{R}{2} + 0.5 - \frac{C}{\cos \alpha/2} Y_\mu \frac{N^\mu + Z^\mu}{1 - N_\mu Z^\mu}. \quad (3)$$

(The minus sign at the denominator of each formula comes from the signature convention we adopt here.) We assume here that the pixel coordinates ranges from 1 to C in along the horizontal direction, number 1 being on the left, and from 1 to R in the vertical direction, from top to bottom. In the usual case where both R and C are even, the center of the screen (i.e., $N^\mu = Z^\mu$) does not correspond to a pixel but to the common corner of the four adjacent central pixels of coordinates $(C/2, R/2)$, $(C/2+1, R/2)$, $(C/2, R/2+1)$ and $(C/2+1, R/2+1)$. We also assume that pixels are perfectly square, i.e, that pixel aspect ratio is exactly 1. In the case where one produces pictures with non square pixels such as for standard video formats (e.g., 576i 4:3, which has a pixel ratio of 12:11), then one has to rescale the second formula (3) along the vertical direction according to the imposed pixel ratio.

The inverse transform that allows to compute the direction N^μ as a function of pixel coordinates can be written in two steps by defining the intermediate quantity W^μ ,

$$W^\mu = Z^\mu + \frac{\cos \alpha/2}{C} \left[\left(i - \frac{C+1}{2} \right) X^\mu + \left(j - \frac{R+1}{2} \right) Y^\mu \right] \quad (4)$$

$$N^\mu = -Z^\mu - 2 \frac{W^\mu}{W_\mu W^\mu}. \quad (5)$$

Now, the way one defines the vectors X^μ , Y^μ , Z^μ is done as follows:

- We start from what we call a reference tetrad that is defined for each event of the spacetime. This tetrad is orthonormal and is made of one timelike vector T^μ and three spacelike vectors R^μ , Θ^μ , Φ^μ , the label of which are of course related to the coordinate system (1) (see later).
- All these vectors are Lorentz transformed according to the unique Lorentz boost Λ_ν^μ that transforms the tetrad timelike vector T^μ into the observer's velocity u_{obs}^μ and leaves invariant any vector orthogonal to both of them.

- The three spacelike vector obtained after performing the Lorentz boost on Θ^μ , Φ^μ , R^μ , i.e., $A^\mu \equiv \Lambda^\mu_\nu \Theta^\nu$, $B^\mu \equiv \Lambda^\mu_\nu \Phi^\nu$, $C^\mu \equiv \Lambda^\mu_\nu R^\nu$ are then rotated by a space rotation R^μ_ν that leaves u^μ_{obs} invariant in order to give the three camera vectors $X^\mu = R^\mu_\nu A^\nu$, $Y^\mu = R^\mu_\nu B^\nu$, $Z^\mu = R^\mu_\nu C^\nu$.

Simple algebra allows to prove that the Λ^μ_ν components are (see, e.g., [24]):

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \frac{1}{1+\gamma}(T^\mu + u^\mu_{\text{obs}})(T_\nu + u_{\text{obs}\nu}) + 2u^\mu_{\text{obs}}T_\nu, \quad (6)$$

where we have defined γ as the Lorentz factor

$$\gamma \equiv T_\mu u^\mu_{\text{obs}}, \quad (7)$$

so that one indeed has $\Lambda^\mu_\nu T^\nu = u^\mu_{\text{obs}}$ and $\Lambda^\mu_\nu(-u^\nu_{\text{obs}} + 2\gamma T^\nu) = T^\mu$, as expected.

Before computing the components of this matrix one of course has to define those of the reference tetrad vectors. A natural choice can be to use a normalized version of the standard spherical Schwarzschild coordinates (i.e., $T^\mu \propto \partial/\partial t$, $R^\mu \propto \partial/\partial r$ and so on), however such a choice is only valid outside the black hole, since the then defined T^μ would no longer be timelike within the black hole. It is therefore more appropriate to define the tetrad that can be associated to a freely falling observer starting from infinity with zero velocity and zero angular momentum. Such an observer four velocity will correspond to the vector T^μ . Then we define R^μ as the unique unit spacelike vector spanned by $\partial/\partial t$ and $\partial/\partial r$ that is orthogonal to T^μ and that reduces to $\partial/\partial r$ at infinity. We keep Θ^μ and Φ^μ unchanged as compared to the first ansatz above.

One then obtain directly the following components

$$T^\mu = \begin{pmatrix} \frac{1}{1-\frac{2M}{r}} \\ -\sqrt{\frac{2M}{r}} \\ 0 \\ 0 \end{pmatrix}, \quad R^\mu = \begin{pmatrix} -\frac{\sqrt{\frac{2M}{r}}}{1-\frac{2M}{r}} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (8)$$

$$\Theta^\mu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix}, \quad \Phi^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r \sin \theta} \end{pmatrix}. \quad (9)$$

This tetrad is defined everywhere, regardless one is inside or outside the black hole (except, of course, at $r = 0$). However, it is not valid everywhere if one explores the full analytical extension of the metric. In this case, if one denotes I our asymptotic region, II the black hole interior beyond the future event horizon (see, e.g., [25]), III the other asymptotic region and IV the region beyond the past event horizon, then the signs of both T^r and R^r must be reversed in region IV and the signs of both T^t and R^t must be reversed in region III.

Regarding the rotation R^μ_ν , we do not compute its components explicitly. Rather, we define its three associated Euler angle. We first rotate A^μ and B^μ by an angle ϕ around C^μ . Then, we rotate the newly obtained A'^μ and

C'^μ by and angle ϑ around the newly obtained B'^μ , and finally, we perform a new rotation by an angle ψ around the new C'^μ . In other words, we process along the following sequence:

$$\begin{aligned} A'^\mu &= \cos \phi A^\mu + \sin \phi B^\mu, & B'^\mu &= \cos \phi B^\mu - \sin \phi A^\mu \\ Z^\mu &= \cos \vartheta C^\mu + \sin \vartheta A'^\mu, & A''^\mu &= \cos \vartheta A'^\mu - \sin \vartheta C^\mu \\ X^\mu &= \cos \psi A''^\mu + \sin \psi B'^\mu, & Y^\mu &= \cos \psi B'^\mu - \sin \psi A''^\mu \end{aligned} \quad (10)$$

III. DRAWING THE CELESTIAL SPHERE – NAIVE VERSION

As stated above, any photon seen originating from unit spacelike direction N^μ orthogonal to the observer velocity u^μ_{obs} is endowed with a wave vector proportional to $k^\mu \propto u_{\text{obs}} - N^\mu$. Once such vector is defined (up to some unimportant proportionality constant), knowing from which direction on the celestial sphere it originates amounts to propagate it backward in time (i.e. backward in its affine parameter p) the geodesic equation

$$\frac{dk^\mu}{dp} + \Gamma^\mu_{\nu\rho} k^\nu k^\rho = 0, \quad (13)$$

where the $\Gamma^\mu_{\nu\rho}$ are the usual Christoffel symbols.

A. First case – Observer outside the horizon

If one works in the usual Schwarzschild spherical coordinates, then this set of equations is written as

$$\frac{dk^t}{dp} = -\frac{A'}{A} k^t k^r, \quad (14)$$

$$\frac{dk^r}{dp} = -\frac{1}{2} A A' (k^t)^2 + \frac{1}{2} \frac{A'}{A} (k^r)^2 + A r ((k^\theta)^2 + \sin^2 \theta (k^\varphi)^2), \quad (15)$$

$$\frac{dk^\theta}{dp} = -\frac{2}{r} k^r k^\varphi + \sin \theta \cos \theta (k^\varphi)^2, \quad (16)$$

$$\frac{dk^\varphi}{dp} = -\frac{2}{r} k^r k^\varphi - 2 \frac{\cos \theta}{\sin \theta} k^\theta k^\varphi, \quad (17)$$

where we have set

$$A \equiv 1 - \frac{2M}{r}, \quad (18)$$

and where the prime denotes a derivative with respect to the r coordinate (i.e., $A' = 2M/r^2$). These equation have of course to be solved together with the position equation:

$$\frac{dx^\mu}{dp} = k^\mu. \quad (19)$$

In practice, this set of equations is solved using an adaptive 4th order Runge-Kutta method inspired from the Numerical Recipes [26]. One step that has to be implemented with some care is the choice of the time step.

As we will see later, two numerically tricky zones lie at $r = 2M$ (horizon crossing) and $r = 3M$ (light circle crossing, where radial motion can be slow and unstable whereas orthoradial motion must be carefully computed; see Appendix). Therefore, the timestep choice is essentially given by the following choices:

- If the geodesic is receding away from the black hole (when propagated backward in time) with r already larger than $4M$, the timestep is chosen proportional to r^2 , so that infinity (or, in practice, a very large value of r/M) is reached after a few steps (typically 4 or 5);
- If the geodesic approaches the black hole, then one chooses a timestep proportional to $r - 4M$;
- If one lies within the $r = 4M$ sphere, then a sufficiently small timestep is chosen so as to insure both stability of the integration and monitoring a possible horizon crossing (see later).

If one is outside the black hole and if we do not consider the maximal analytic extension of the metric, then this equation has to be solved only when the geodesics originates from infinity. By setting the constants of motion E and L^2 by their standard definition, i.e.,

$$E \equiv \pi_t = g_{t\mu} k^\mu = A k^t, \quad (20)$$

$$L^2 \equiv r^4 ((k^\theta)^2 + \sin^2 \theta (k^\varphi)^2), \quad (21)$$

then the geodesic equation needs to be computed if and only if

1. $L^2/E^2 \geq 27M^2$ and $r > 3M$, or
2. $L^2/E^2 \leq 27M^2$ and $k^r < 0$.

(Since this is a fairly well-known result we just recall it here, but for the sake of completeness derive it in Appendix.)

If none of these conditions are satisfied, this means that the geodesic originates from the past event horizon, or, from an astrophysically realistic point of view, from the infinitely redshifted surface of the collapsing object which gave birth to the black hole as it was passing through the horizon. If one does not work within the maximal analytic extension of the metric, then such geodesics do not carry any photon and the corresponding pixel is black. If, on the contrary, one works in the maximal analytic extension, then the geodesic can be propagated back to the past singularity and imaged provided one decides of some emission properties of the past singularity.

For geodesics originating from infinity, one obtains at the end of integration a wavevector k_∞^μ whose only non negligible components are k_∞^t and k_∞^r , the two others tending to 0 when r tends to infinity because of angular momentum conservation. Regarding the position, r and t both tend to minus infinity with their difference $r - t$ being almost constant, and θ and φ tend to be constant. This is because as long as the radial coordinate r is much

larger than the impact parameter $b \equiv L/E$, the geodesic can be considered as (almost) purely radial and originating from the direction defined by the above mentioned θ and φ .

In addition, one can compute the redshift z of the photons we receive. This is done through the standard formula

$$1 + z = \frac{k_\infty^t}{k^\mu u_{\text{obs } \mu}} = \frac{E}{k^\mu u_{\text{obs } \mu}}, \quad (22)$$

where the numerator is evaluated (as the subscript indicates) at infinity, whereas the denominator is computed at observer's position before integration. Note that this formula include both the kinetic and gravitational redshift.

Once the initial direction of the photon and the redshift are known, we can draw the corresponding pixel.

B. Second case – Observer inside the horizon and/or within the maximal analytic extension

The set (14–17) is valid only when geodesics do not cross the horizon. Therefore, if the observer is within the horizon, it is not possible to simulate what he/she sees of the celestial sphere using these equations. In order to do so, one has to use another system of coordinates, the most natural of which being that of Kruskal. In this case, the subset of coordinates (t, r) have to be replaced by the subset (U, V) defined by the following procedure. First, we define the so-called “tortoise” coordinate r^* by

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right|, \quad (23)$$

so that

$$dr^* = \frac{dr}{1 - \frac{2M}{r}}. \quad (24)$$

Here, r^* is a growing function of r outside the horizon and a decreasing function of r inside (regardless one considers the maximal analytic extension or not). Then, we define u and v such as

$$u \equiv t + r^*, \quad (25)$$

$$v \equiv t - r^*, \quad (26)$$

and finally U and V are defined through

$$U \equiv \epsilon \exp \left(+ \frac{u}{4M} \right), \quad (27)$$

$$V \equiv \eta \exp \left(- \frac{v}{4M} \right). \quad (28)$$

The constant $\epsilon, \eta = \pm 1$ are then chosen so that both U and V grow with any observer proper time or grow with any null geodesics affine parameter when it grows as the geodesic is propagated in the future. Their value are summarized in Table I.

Region	Remark	ϵ	η
I	t grows with proper time	1	-1
II	r decreases with proper time	1	1
III	t decreases with proper time	-1	1
IV	r grows with proper time	-1	-1

TABLE I. Values of the parameters ϵ and η as defined in Eqns (27–28). Their value allow the null coordinates U and V to with proper time regardless the observer position within the maximal extension of the Schwarzschild metric. Which of the variable t or r of the standard Schwarzschild coordinates that is bound to grow or decrease with proper time from any timelike geodesics is also given.

This choice for ϵ and η also possesses the nice property to ensure that the product UV can be expressed as a function of r only, without any reference to ϵ or η ¹:

$$UV = -\exp\left(\frac{r}{2M}\right) \frac{r-2M}{2M}. \quad (29)$$

This being set, the geodesic equation (14–17) has then to be replaced by the following set of equations

$$\ddot{U} = -\frac{\partial_U F}{F} \dot{U}^2 - \frac{r}{F} \partial_U r \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right), \quad (30)$$

$$\ddot{V} = -\frac{\partial_V F}{F} \dot{V}^2 - \frac{r}{F} \partial_V r \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right), \quad (31)$$

$$\ddot{\theta} = \cos \theta \sin \theta \dot{\varphi}^2 - \frac{2\dot{\theta}}{r} (\dot{U} \partial_U r + \dot{V} \partial_V r), \quad (32)$$

$$\ddot{\varphi} = -2 \frac{\cos \theta}{\sin \theta} \dot{\theta} \dot{\varphi} - \frac{2\dot{\varphi}}{r} (\dot{U} \partial_U r + \dot{V} \partial_V r), \quad (33)$$

$$(34)$$

where we have set

$$F \equiv g_{UV} = -\frac{8M^2 A}{UV}, \quad (35)$$

A being defined in Eq. (18) and where we have used the following intermediate quantities

$$\partial_U r = \frac{2MA}{U}, \quad (36)$$

$$\partial_V r = \frac{2MA}{V}, \quad (37)$$

$$\partial_U F = \frac{2MAF'}{U}, \quad (38)$$

$$\partial_V F = \frac{2MAF'}{V}, \quad (39)$$

in which F' denotes the derivative of F with respect to r , which is quite straightforwardly deduced from Eq. (29) and can be written as

$$F' = \left(-\frac{1}{2M} + A'(r) \right) \frac{F}{A}. \quad (40)$$

¹ This is because the product $\epsilon\eta$ has the same sign as $2M - r$ so that it compensate to absolute value that arises in the definition of r^* in Eq. (23)

Since F' only appears when multiplied by A the set of equation is regular at horizon crossing.

The set of variables U, V is well suited for horizon crossing and is in principle defined everywhere on the maximal analytic extension of the manifold. In practice it is however not possible to use it everywhere since the exponential dependence of both U and V in term of r and t make it numerically impossible to use as soon as one goes a few Schwarzschild radii away from the black hole. Therefore, we adopt the following procedure in order to choose the coordinate system:

- From the knowledge of constants E and L^2 and position, we determine whether the geodesic we are interested in has any chance to cross horizon;
- If not, then we use the Schwarzschild coordinates r, t ;
- If so, then we check whether one is close to the horizon;
- If not, then we keep the Schwarzschild coordinates;
- If so, then we switch to Kruskal coordinates;
- Then we keep on using Kruskal coordinates till horizon has been crossed, in which case we switch back to Schwarzschild coordinates.

The only arbitrariness here lies in where exactly we decide to switch from one coordinate system to the other. In practice, the radial motion of photon is determined by and effective potential $V(r)$ given in Eq. (A.3). This potential show that in some case, a geodesic may spend a large amount of time around $r = 3M$ (the so-called light circle) around which the one dimensional radial motion is unstable. Therefore, we choose to impose that the light circle crossing is made using Schwarzschild coordinates r, t .

C. From geodesic equation solution to RGB coordinates

In a perfectly realistic situation, we would have at our disposal a spectral map of the celestial sphere, i.e., spectroscopic data for each direction of it. Then we would modify the spectrum according to the computed redshift and then compute the eye response to that observed spectrum and then deduce the corresponding RGB coordinates of the pixel. In practice, such a precision is rarely taken into account. Many papers focus on the aspect of an accretion disk and do not care about the celestial sphere itself, while others care only with stars. In practice, when one wants to include a celestial sphere as a background image distorted but the black hole, one often uses a pixellized version from full sky picture of our sky, whether realistic or computer generated, but which is does not include any spectral information (apart from its color). This therefore gives the correct colors of the

sky seen by the observer only when redshift is negligible as such procedure does not allow to compute the color or intensity change of the celestial sphere due to the redshift is not possible but this can be compensated by various visual artifacts. On the contrary, it is possible to define by hand the spectral information for each direction of the celestial sphere. This allows to have a perfect rendering of the influence of the redshift, but possibly at the cost of performing a less aesthetic rendering of the sky. In what follows, we shall use both approaches.

IV. DRAWING THE CELESTIAL SPHERE – SOPHISTICATED VERSION

In the Section above, the number of geodesic equations we have to solve is equal to the number of pixels of the screen and can therefore easily reach several millions and severely affect the computational time for each picture. The situation is worsened by the fact that some smoothing may be necessary when computing the image, i.e., one may need to split one pixel into several subpixels, compute the color of all of them and average the result accordingly.

Such a large number of integrations of the geodesic equation is actually not necessary because, as is well known, null geodesics are described by a single parameter, which can be taken to be the impact parameter $b = L/E$. Moreover, we are considering a situation where the only information about the geodesics we are interested in is the amount of deflection the photon trajectory experiences between the direction it travels at the observer's position and the direction it was traveling at infinity. Therefore, we adopt the following procedure:

- Before computing the image, we solve the geodesics equation for a freely falling observer (the one that defines our reference tetrad, with four-velocity T^μ) for all possible angles δ between the geodesic and the radial direction. This amounts to sample, in a more appropriate way, all the values of the geodesic impact parameter.
- For each of these geodesics we obtain the angle $\delta - \chi$ by which the geodesics has been deflected.
- We sample this quantity sufficiently by computing a moderately large number of geodesics so that the both functions $\chi(\delta)$ and its inverse $\delta(\chi)$ are well sampled.
- Then, for each pixel of the image, compute the angle δ between the corresponding null geodesic and the radial direction, and identify the plane into which the photon trajectory lies.
- We rotate within this plane the observer position by an angle $\chi + \pi$ so as to identify the point of the celestial sphere the photon originates from.

- We compute the RGB coordinates of the pixel one had to draw following the selected assumptions for redshift rendering.

The angle δ we use here is defined with respect to a freely falling observer. This means that it is given by the formula

$$\cos \delta = - \left(\frac{k^\mu}{k^\nu T_\nu} - T^\mu \right) R_\mu, \quad (41)$$

or, equivalently, this means that we will integrate a bunch of geodesics whose wavevector k^μ is defined as

$$k^\mu \propto T^\mu + \cos \delta R^\mu + \sin \delta (\cos \varpi \Phi^\mu + \sin \varpi \Theta^\mu), \quad (42)$$

with an appropriate sampling of δ . With this definition, a radial geodesic originating from the black hole past horizon is described by an angle $\delta = 0$, and a radial geodesic originating from infinity is described by $\delta = \pi$. Because of spherical symmetry, when we compute the $\chi(\delta)$ function, we can place ourselves in the equatorial plane at the same r coordinate as the observer and by considering equatorial geodesics only that start from the x axis (i.e., our freely falling fiducial observer lies at coordinates $(t, r, \pi/2, 0)$ and $\varpi = 0$ in Eq. (42)). From this position and this wavevector, we propagate the geodesic backward in time till it reaches very large values of r (typically larger than $10^{10}M$). We then obtain the azimuthal angle φ_∞ with respect to the observer of the corresponding origin of the geodesic (for example, $\varphi_\infty = 0$ for $\delta = \pi$, and the angle χ is then given by

$$\chi = \phi_\infty - \pi. \quad (43)$$

Should there be no aberration nor deflection of light, then one would just have $\chi = \delta$ regardless of the observer position and velocity². However this is not what happens here and the deflection experienced by any geodesic between its starting point at infinity and the moment it crosses the fiducial observer world line given by $\delta - \chi$ which is zero only when $\delta = \pi$. An example of deviation function $\chi(\delta)$ (or, in fact, $\phi_\infty(\delta)$) is given in Figure 1.

Note that the two angles δ and χ are associated to somewhat different contexts: δ is an angle defined by a freely falling observer in the Schwarzschild metric, whereas χ corresponds to an angle measured by a static observer in Minkowski space whose origin and orientation matches that of the Schwarzschild metric.

In practice, sampling this $\chi(\delta)$ function necessitates a few thousands of geodesics to be computed, i.e. a factor between 10^2 and 10^3 less than the brute force computation of the previous section. This situation is also enhanced by the fact that it suffices to perform the computation in the equatorial plane, so that, in practice, there

² We implicitly neglect any parallax effect here.

V. DRAWING THE STARS

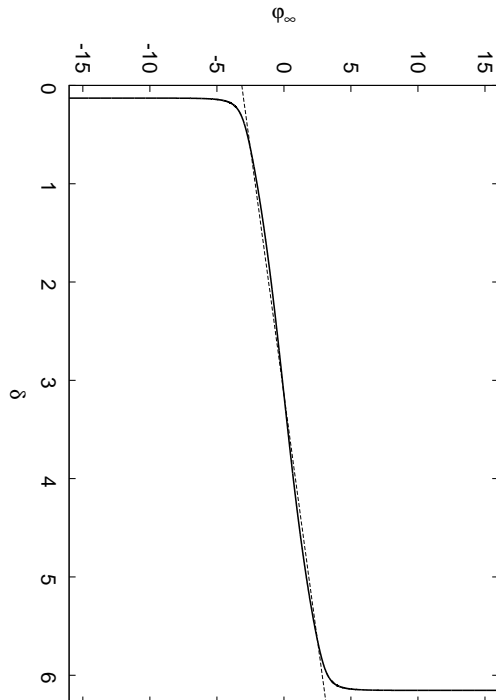


FIG. 1. An example of the deviation function $\varphi_\infty(\delta)$. The one shown here was computed for a freely falling observer on a Schwarzschild black hole at coordinate distance $r = 30M$. In the absence of any relativistic effect, the function $\varphi_\infty(\delta)$ would trivially reduce to $\varphi_\infty = \delta - \pi$, shown in dashed lines. But the adjunction of aberration (since the observer is freely falling) and light deflection modify it, only slightly away from the black hole (δ close to π), or much more importantly toward the black hole (δ close to 0 or 2π). The black hole angular radius is given by the first value of δ for which φ_∞ diverges. Here, the function was sampled so as to include values of $|\varphi_\infty|$ slightly larger than 5π , which is sufficient to include any visible ghost images of stars in most situations.

is no need to consider the θ and k^θ variable in the ODE system (14–17,19).

Then, once this deflection function is computed, all what we need to do rely on very simple trigonometric operations for all each pixel of the screen and some search in the deflection function $\chi(\delta)$ pre-computed array. More importantly, the deviation function will further be used in the next section (through its inverse) in order to perform a very clean rendering of the stars.

The main drawback of the sky rendering above is that it is not well suited to include point sources. A point source such as a star will never appear as perfectly point-like. Because of diffraction a point source will always appears as possessing a small but non zero angular size typically of circular shape (and possibly diffraction patterns) and can, in principle, belong to the pixellized version of the celestial sphere. However, when distorting the image of such a source, its appearance will also be distorted. As is well known, amplification due to gravitational lensing goes with a large amount of shear, so that an initially circular pattern will become quite elongated. This is not the way this source should appear because given the actual smallness of a star angular size, even in a strong lensing regime it should be considered a point-like and its visual finite angular size would only result in the optical distortions caused by the observation apparatus. Consequently, it is not possible to consider stars as “points” that would be “impainted” on the celestial sphere, and stars (or any almost pointlike sources) should be processed using a different procedure.

A. Direct raytracing from interpolation

Since we work in a Schwarzschild metric, any direction on the celestial sphere will possess an infinity of images seen from any observer point of view, even though most of these images will be extremely faint and close to the edge of the black hole silhouette. Since the metric is spherically symmetric, any geodesic is planar and all the geodesics starting from the same point of the celestial sphere and reaching the observer belong to the same plane.

What we initially know about a star is its position \mathbf{n}^* on the celestial sphere. We also know the observer position, and consequently we know the plane spanned by the observer, the black hole and the star, as well as the angle between the observer radial position and and star position, ϕ_∞^* . We can choose the orientation of the star-observer-black hole plane so that ϕ_∞^* lies between 0 (observer is between the star and the black hole) and π (black hole is between the observer and the star). We can now invert the procedure outlined in the previous section, with the extra complication that for a given δ_0 such that $\phi_\infty(\delta_0) = \phi_\infty^*$, there exists other values of δ that we shall note δ_k such that

$$\phi_\infty(\delta_k) = \phi_\infty^* + 2k\pi. \quad (44)$$

Those are the ghost images of the star. There exists an infinity of such images as long as the function $\phi_\infty(\delta)$ diverges, which is the case for the Schwarzschild metric. These ghost images are not difficult to find numerically as long as the function $\phi_\infty(\delta)$ is accurately sampled. We therefore can find without difficulty the apparent position of any ghost image of any star.

B. Amplification

This being done, we know the direction under which we see a given image of the star. However, because of both aberration and lensing, the actual (microscopic) angular size of the star will be different, and hence its luminosity. In order to take account for this, we shall define two directions very close to that where the star (or, in fact, one of its image) is seen. Since we work here in the observer's frames, the directions we are talking about can be expanded in term of the spacelike vectors X^μ, Y^μ, Z^μ and can be considered as Euclidean three-vectors which we shall write in bold notations and the direction into which the (image of the) star is seen will be written \mathbf{n} . In order to define two directions infinitesimally close to \mathbf{n} , we choose one direction \mathbf{n}' that is different from \mathbf{n} . This can be for example either \mathbf{x} or \mathbf{z} (one may choose the one among these two which has the smallest dot product with \mathbf{n}). Then, we compute

$$\mathbf{n}_1^\perp = \frac{\mathbf{n} \wedge \mathbf{n}'}{|\mathbf{n} \wedge \mathbf{n}'|}, \quad (45)$$

and then

$$\mathbf{n}_2^\perp = \mathbf{n} \wedge \mathbf{n}_1^\perp. \quad (46)$$

We then choose two small quantities δ_1 and δ_2 and define

$$\mathbf{n}_1 = \mathbf{n} + \delta_1 \mathbf{n}_1^\perp = \mathbf{n} + \delta_1 \mathbf{n}_1^\perp, \quad (47)$$

$$\mathbf{n}_2 = \mathbf{n} + \delta_2 \mathbf{n}_2^\perp = \mathbf{n} + \delta_2 \mathbf{n}_2^\perp. \quad (48)$$

Up to $O(\delta_{1,2}^2)$ terms, these two vectors are units vectors that are very close to \mathbf{n} . Now, it is obvious that the solid angle Ω spanned by three directions $\mathbf{n}, \mathbf{n}_1 = \mathbf{n} + \delta_1 \mathbf{n}_1^\perp$ and $\mathbf{n}_2 = \mathbf{n} + \delta_2 \mathbf{n}_2^\perp$ is given by the formula

$$\Omega = \mathbf{n} \cdot (\delta_1 \mathbf{n}_1^\perp \wedge \delta_2 \mathbf{n}_2^\perp) = \mathbf{n} \cdot (\mathbf{n}_1 \wedge \mathbf{n}_2). \quad (49)$$

Equivalently, if we propagate the null geodesics which originate from these three direction till the celestial sphere, we obtain three direction on the celestial sphere, $\mathbf{n}^*, \mathbf{n}_1^*$ and \mathbf{n}_2^* which span a solid angle Ω^* given by

$$\Omega^* = \mathbf{n}^* \cdot (\mathbf{n}_1^* \wedge \mathbf{n}_2^*). \quad (50)$$

With these notations, the amplification or desamplification factor f induced both by lensing and aberration is simply written as

$$f = \frac{\Omega}{\Omega^*}. \quad (51)$$

Computing this factor numerically is in fact not necessary. There exists an analytical formula for amplification due to aberration, and the deflexion function $\chi(\delta)$ (or, in fact, the derivative of its inverse) allows after some algebra to address the lensing part. However in practice this does not allow to perform significant enhancement in term of CPU time so that we won't address this here but rather keep it for future work.

C. Effective drawing of the star

In the two last subsections, we explained how to determine the position of (the image of) a star on the observer's screen, in addition to its redshift and the amplification of its light because of aberration and lensing. For simplicity, we shall assume here that stars have a black-body type emission whose temperature is given by their spectral type (O and A being hotter, K and M cooler).

If one wants to compute the colour perceived by human eye a given light source (in term of, say, its RGB coordinate of a computer screen), one has to know the eye perception for each visible monochromatic frequency. These data are called the spectral tristimulus values are have been tabulated since a long time by the dedicated authority, the International Commission on Illumination (CIE) [27]. For simplicity, we assume that eye response do not depend on light intensity, that is, we questionably assume that the observer's vision always works in diurnal (photopic) mode rather than in low brightness (scotopic) environment. This allows more colorful hues for stars than what we are used to. In any case, if one starts from a light source with a given spectrum, one can compute the RGB coordinates of this source by using the trichromatic tables delivered by the CIE.

In practice, with the assumptions we make about the star spectra, the procedure is the following:

1. Prior to launching the code, we have processed our star catalog in order to transform each star's spectral type and magnitude in the V-band into a bolometric magnitude m_* and a surface temperature T_* .
2. For each image of each star, we compute the corresponding redshift z and amplification factor f .
3. We deduce that the star image possesses an apparent (in the sense of observer-dependent) bolometric magnitude and temperature given by

$$m_*^{\text{obs}} = m_* + 4 \log(1+z) - \log f, \quad (52)$$

$$T_*^{\text{obs}} = \frac{T_*}{1+z} \quad (53)$$

4. With these new values, we compute the RGB coordinates of this image of the star.

This being done, we have to decide how to simulate a pointlike light source with these RGB coordinates. Since the source is supposed to be pointlike, the most natural choice is to add to the pixel where the star image is seen the RGB coordinates of the star to that of the already computed background. However, this naive assumption quickly leads to difficulties. The reason is that there is nothing that guarantees that the RGB coordinates are smaller than 1, i.e., that adding the star to this pixel will not saturate it. If they are not, putting all the luminosity

of the star into a single pixel will truncate its true luminosity to the maximum that a pixel can draw and many bright star will have, in practice, identical magnitude because of the limitations of a computer screen. Therefore, in order to allow recovering the whole luminosity range of observable stars, we draw them as extended blobs which are several pixel wide. We have found that a pleasant rendering is obtained if our blob intensity profile looks like a truncated Gaussians both in R, G, and B colors. This means that we impaint the already computed distorted celestial sphere by those blobs, imposing that the Gaussians are centered on the actual position of the star (not necessarily at the center of the pixel they belong to) and by choosing their size so that their integrated luminosity corresponds to the one that has been computed. In other words, the brighter the star, the larger the blob that represents it. This procedure is inspired by the beautiful pictures made by famous amateur astronomer Akira Fuji (see, e.g., [28]) which are obtained by putting a diffusing filter in front of the camera so as to artificially spread a bright star images on some extended area of the picture in order to more faithfully reproduce the luminosity contrast between faint and bright stars.

In order to reproduce the most satisfactory rendering of the stars, some cooking is necessary here. For example, it appears less artificial to saturate the centre of the blobs that represent bright stars, so that its color is less saturated than the edge of the blob which reproduces more faithfully the star color. Also, it appears that when performing videos, a faint stars slowly crossing the screen appears more esthetic if one imposes that its blob is always at least a few pixel wide in diameter even if the center of the blob is then not saturated.

Even if there are obviously some “artistic” choices that are made here (apologizing in advance that not every reader will agree with them!), we insist on the fact that the really physically significant quantities that are needed, m_{\star}^{obs} and T_{\star}^{obs} are computed as accurately as possible, so that we have very carefully splitted the problem into its physical content (m_{\star}^{obs} and T_{\star}^{obs}) and its representational content (how to associate a colored blob to these two quantities).

VI. DATA THAT ARE ACTUALLY USED TO PRODUCE IMAGES

A. Background sky

For pedagogical purposes, a celestial sphere made of a coordinate grid is by far sufficient. In what follows we have defined a celestial sphere under the form of a checkerboard structure of 5 degrees both in latitude and longitude. The two type squares (“light”) and (“dark”) all correspond to blackbody type emission but with same temperature (8000 K) but different intensity (a factor 4 between light and dark squares). In order to avoid thinner and almost triangular squares toward the pole, the

polar regions are covered by discs of 5 degrees in diameter which are both redder ($T = 2000$ K) and brighter.

For aesthetic rendering and/or astronomical outreach use, it is by far better to use a celestial sphere that looks like a real one. The simple way to obtain this is by using actual pictures of the sky seen from Earth. Many such high resolution pictures of this exist. One of the most famous is the one made by A. Mellinger some time ago [29], at the same epoch as the one made by S. Brunier [30]. Unsurprisingly, the latter was much advertised in French speaking countries, whereas the former was mostly known in the rest of the world. Independently however of the astonishing quality of these two pictures, both suffer from the fact that stars are part of the pictures in the sense that their appear to be impainted on the celestial sphere. This is not satisfactory for the reasons given in Sec. V.

For this reason, it is better to look for a full sky picture of the celestial sphere whose star have been removed. Whether or not this could be done starting from these existing and highly star crowded pictures is not known to the author, however this was achieved in a rather clean way by the 2MASS collaboration which produced a starless picture of the celestial sphere [31]. It seems that the pictures is not actually “starless”, but that the luminosity of each pixel was computed by averaging the luminosity of the stars over some windows function. The picture obtained in this way is of moderate size (2400×4700 pixels) corresponding (once borders are removed) to a resolution of around $4.8'$ per pixel. This is a factor ~ 5 coarser than the natural resolution of human eye, however, for practical purposes, what is of interest is the ratio between the digital resolution of the produced images and that of the images that are used for rendering. If one considers normal (for modern screen standards) pictures of around 1280 pixel wide corresponding to a view computed with an opening angle of 90 degrees, then the theoretical resolution of such a picture is around $4.2'$ which is only marginally better than the 2MASS picture, which is therefore is sufficient for many purposes.

This picture suffers however from small but noticeable problems. The most obvious one is that the lower border is missing on a width of a few pixels, so that one has to complete the missing part by some (rather arbitrary) procedure. The second one is that the 2MASS project has observed the sky in the infrared bands and the structure of the Milky way is somewhat affected by this. The most noticeable difference comes from the fact that there is far less absorption, especially in the direction of the Galactic centre which appears much more regular and more symmetric with respect to the Galactic plane than in optical images. Also, since this picture is unavoidably made in false colors, the overall hue does not correspond to visible image, notable the Milky way band which is both brighter and has much more yellowish hue than what human eye is used to. Finally, since the image is already given in term of a planar projection and its pixellisation

follows this projection³. When viewed in a spherical context, then the underlying pixellisation of the image appears more or less elongated depending on the direction of observation. This is particularly true at the joining of the two edges of the pictures some quite visible and unaesthetic patterns are visible on the bare image.

B. Stars

Regarding the stars, visual inspection of simulated images shows that the more stars, the more spectacular the result is. We therefore need a large, uniform and magnitude limited catalog comprising at least several 10^4 stars.

We have chosen to use the electronic version of the Henry Draper catalogue with its extension [33], that were initially published by A. J. Cannon and E. W. Pickering between 1918 and 1924. This catalog comprises more than 250,000 stars. For high magnitudes, it is not uniform, some regions of the sky having a deeper coverage (the Galactic anticenter, among others). We therefore truncate up to some magnitude around 9. After this, some hand made change have to be performed because the magnitude of some binary star systems are not given and this happens for some easily noticeable stars such as β Lyrae.

For very high resolution images, we also have used the ASCC catalogue of around 2.5 million stars [34].

C. CPU issues

With the described implementation, the typical computational time for a single high resolution image is split in equal proportion between the few thousand geodesic calculation sampling the deflection function $\chi(\delta)$, and drawing the stars pixel by pixel along the lines described above. Typical images of one million pixel initially computed at a twice larger resolution and then smoothed needed in the first versions of our code needed one minute of single CPU time to be computed. Assuming one makes a movies at 25 of 30 frames per second, one second of the movie can be computed in less than half an hour, thus allowing to obtain a one minute long movie in less than one CPU day.

In some situations however, most notably when large portions of the sky experience a large blueshift (for example, a freely falling observer within the horizon and approaching the singularity or static observer close to the horizon), the CPU time is dominated by the star drawing, something which is far from being optimized, thus reducing the length of the movie that could be made in one

CPU day. However, we did not meet any critical CPU issues here. Moreover, since any frame can be computed independently of the others, we did not have any need of parallelization. It was in practice simpler to launch by hand our code to compute a few hundreds of frame per computer, and then to split the not so numerous hard-to-compute frames after they had been identified.

VII. SOME IMAGES

The methods that we have outlined can now be implemented to compute high quality images of black holes in various situations, and to explore the unexpected variety of special and general relativistic effects that one would visually experience close to a black hole. For pedagogical purpose, we shall try to isolate each of those effects one by one.

A. Special relativistic effects only

In this subsection, we shall consider Minkowski space only.

Let us assume an observer traveling at a constant velocity \mathbf{v} with respect to a reference observer whose celestial sphere is well-defined. We note (t, \mathbf{x}) a set of Cartesian coordinates for the reference observer and (t', \mathbf{x}') those of the second one. Up to some unimportant constant, the second set of coordinates is expressed in term of the first one as:

$$t' = \gamma(t - \mathbf{x} \cdot \mathbf{v}), \quad (54)$$

$$\mathbf{x}'_{\perp} = \mathbf{x}_{\perp}, \quad (55)$$

$$\mathbf{x}'_{\parallel} = \gamma(\mathbf{x}_{\parallel} - \mathbf{v}t), \quad (56)$$

where \mathbf{x}_{\parallel} and \mathbf{x}_{\perp} represent the parallel part and the perpendicular part of the spacelike coordinates with respect to velocity \mathbf{v} . As well-known, the last two equations can be rewritten into a single one:

$$\mathbf{x}' = \mathbf{x} - \gamma \mathbf{v}t + \frac{\gamma^2}{\gamma + 1}(\mathbf{v} \cdot \mathbf{x})\mathbf{v}. \quad (57)$$

The associated Lorentz transform can be seen as the matrix whose components are $\Lambda^{\mu}_{\nu} = \partial x'^{\mu} / \partial x^{\nu}$, which here can be written (see Eq. ([?]))

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma \mathbf{v} \\ -\gamma \mathbf{v} & \text{Id} + \frac{\gamma^2}{\gamma + 1} \mathbf{v} \otimes \mathbf{v} \end{pmatrix}. \quad (58)$$

From the point of view of the reference observer, a wavevector k^{μ} associated to a direction \mathbf{n} has components given by

$$k^{\mu} = \omega \begin{pmatrix} 1 \\ -\mathbf{n} \end{pmatrix}. \quad (59)$$

³ Regarding this, it has to be noted that the 2MASS website claims that it is an Aitoff projection but following the nomenclature of Ref. [32], it rather seems to be an Aitoff-Hammer projection.

In the second observer frame, the wavevector has the new coordinates

$$\begin{aligned} k'^{\mu} &= \Lambda^{\mu}_{\nu} k^{\nu} \\ &= \omega \begin{pmatrix} \gamma(1 + \mathbf{v} \cdot \mathbf{n}) \\ -\mathbf{n} - \gamma \mathbf{v} \left(1 + \frac{\gamma}{\gamma+1} (\mathbf{v} \cdot \mathbf{n})\right) \end{pmatrix} \\ &\equiv \omega' \begin{pmatrix} 1 \\ -\mathbf{n}' \end{pmatrix}. \end{aligned} \quad (60)$$

The direction \mathbf{n}' and frequency ω' of the corresponding photon seen from the second observer are therefore

$$\mathbf{n}' = \frac{\mathbf{n} + \gamma \mathbf{v} \left(1 + \frac{\gamma}{\gamma+1} (\mathbf{v} \cdot \mathbf{n})\right)}{\gamma(1 + \mathbf{v} \cdot \mathbf{n})}, \quad (61)$$

$$\omega' = \omega \gamma (1 + \mathbf{v} \cdot \mathbf{n}). \quad (62)$$

Although it is not obvious at first sight, the vector \mathbf{n}' has norm 1 as expected.

1. Aberration

Aberration describes the way the spacelike components of a null vector are transformed during a Lorentz boost. Its main effect is that the angle between some star and the direction one is travelling to diminishes as velocity increases. Starting from Eq. (61) and denoting α , α' the angle between \mathbf{n} and \mathbf{v} on the one hand, and \mathbf{n}' and \mathbf{v} on the other hand, one has

$$\cos \alpha' = \frac{\cos \alpha + v}{1 + v \cos \alpha}, \quad (63)$$

so that, as expected,

$$\cos \alpha' - \cos \alpha = \frac{v \sin^2 \alpha}{1 + v \cos \alpha} > 0. \quad (64)$$

The relation between α and α' can be inverted, either by direct algebra or by noticing that it suffices to change the sign of v in order to make the inversion make sense, so that

$$\cos \alpha = \frac{\cos \alpha' - v}{1 - v \cos \alpha'}. \quad (65)$$

A few examples of the aberration effect are shown in Figures 2,3.

2. Doppler

The frequency shift ω' given by Eq. (62) is more conveniently written in term of the angle α' as it is the one that is actually observed. It reads

$$\omega' = \frac{\omega}{\gamma(1 - v \cos \alpha')}. \quad (66)$$

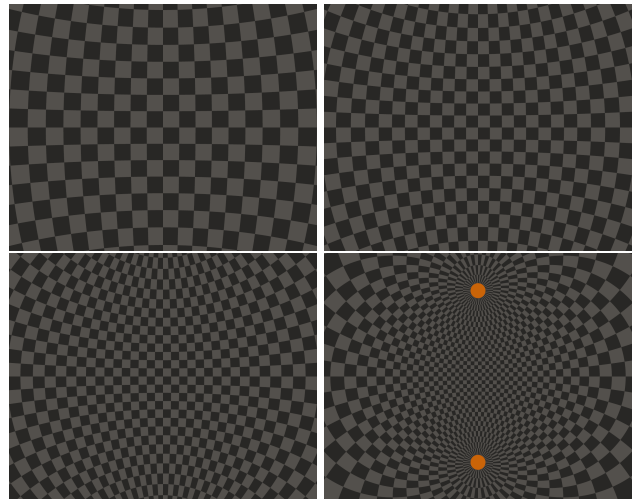


FIG. 2. Aberration seen by an observer starting from a static situation with respect to the celestial sphere (upper left image) to $v = 0.3$ (upper right image), $v = 0.6$ (lower left image), and $v = 0.9$ (lower right image). In all the pictures the field of view is 90 degrees along the horizontal direction. The “squares” delineating the celestial sphere are 5 degrees wide both in latitude and longitude (see Section VIA). Eq. (65) can be checked by visual inspection of the number of squares along the central horizontal band. The initial angular separation between the central part of the vertical edges of the images is 90, 117.8, 158.6, 244.1 degrees, respectively. The two poles of the celestial sphere, which by definition are 180 degrees apart for a static observer are now, from Eq. (63), only 51.7 degrees apart on the last image.

One recovers the usual result that the frequency shift goes between $\sqrt{(1-v)/(1+v)}$ and $\sqrt{(1+v)/(1-v)}$ when going from the opposite direction to the direction of \mathbf{v} . Along the perpendicular direction (i.e., $\cos \alpha = 0$), the frequency shift is γ^{-1} , i.e., there is an observed redshift. The region along which there is a blueshift if the one where $\omega' > \omega$, which corresponds to

$$\cos \alpha' > \frac{\gamma v}{\gamma + 1}, \quad (67)$$

which, in term of the angle α corresponds to

$$\cos \alpha > -\frac{\gamma}{\gamma + 1}. \quad (68)$$

This means that when the velocity is large, the angular size of the blueshifted region is increasingly smaller (the lower bound on $\cos \alpha'$ increases), but corresponds to an initially increasingly larger patch of the sky seen by the first observer (the lower bound on $\cos \alpha$ increases). An example of the observed color change due to the Doppler effect is shown in Fig. 4.

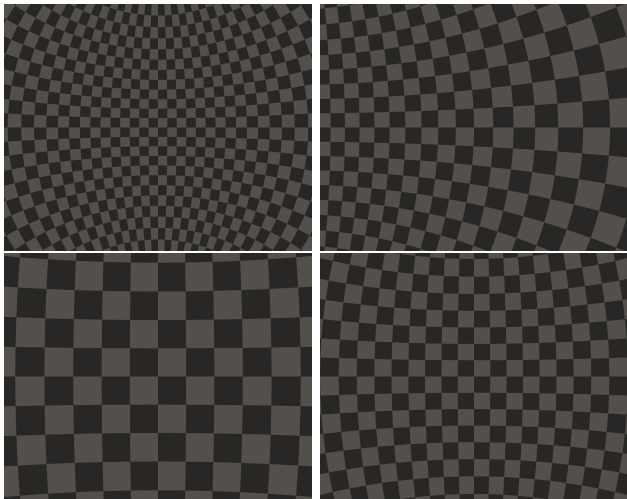


FIG. 3. Aberration seen by an observer travelling at $v = 0.5$ with respect to the frame where the celestial sphere is defined (see Sec. VI A). Upper left image shows the front view, followed by the right view (upper right) and rear view (lower left). Without aberration, these three views should match the static one, in the lower right corner.

3. Intensity

In addition to the frequency shift, the Doppler effect produces a variation of the overall intensity of a light source. In the case considered here, where we assume that our celestial sphere (and, later, the stars) have a blackbody emission, the bolometric luminosity varies as T^4 , where T corresponds to the temperature of the celestial sphere pixel or star of interest. Consequently, the bolometric luminosity is modulated by a factor $(1+z)^{-4}$, where the redshift z is given by

$$1+z = \frac{\omega}{\omega'}. \quad (69)$$

The intensity variation therefore varies of a factor $[(1+v)/(1-v)]^4$ between the front and rear directions, a factor which can be quite large for relativistic velocities ($\sim 130\,000$ for $v = 0.9$), which make it difficult to represent on a computer screen since with sRGB coordinates, the relative intensity between the brightest pixel and the dimmest one (for a fixed hue) is $12.92 \times 255 = 3294.6$. Note however that such a factor is valid for the bolometric luminosity only. Taking into account the eye response drastically change this factor, although it does not improve much the situation: a highly redshifted object become completely invisible not because its bolometric luminosity decreases, to the $(1+z)^{-4}$ factor but because almost all its energy becomes radiated in the infrared domain which is not visible at all to a human eye. An example of the intensity effect is shown in Fig. 5.

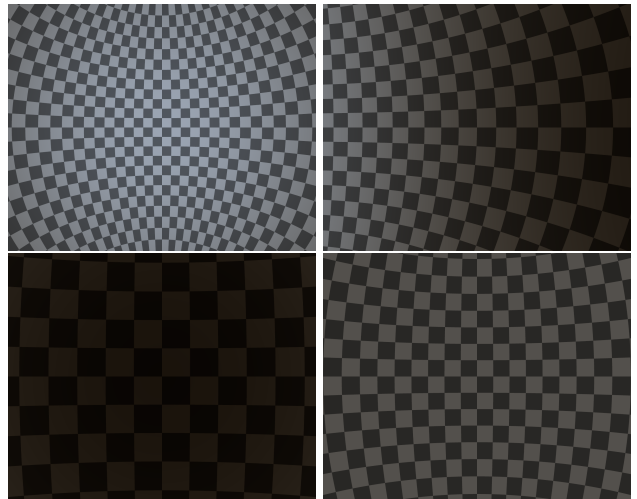


FIG. 4. Same as Fig. 3 above, but now including color change due to Doppler shift. The bolometric intensity of each pixel of the celestial sphere is kept unchanged, so that the intensity changes that are seen are due to the variation of sensitivity of the eye with respect to the variable temperature, the loss of sensitivity being largest in the rear direction, where the red spectrum peaks in the infrared domain and has an intensity in the visible band that is exponentially suppressed as redshift increases.

4. Amplification

When considering our checkerboard-like celestial sphere, the total intensity of a given square relies on the combination of its temperature change because of the Doppler effect and its angular size change because of aberration. This last part is computed implicitly by the fact that the number of pixels which span the square changes because of aberration. If we consider a star, then only the Doppler term is known a priori. However, the star is an extended object, although a tiny one, just as our checkerboard is. Therefore, we must add to its intensity change the amplification factor whose computation is outlined in Section [?].

B. General relativistic effects only

1. Light bending

Let us now add a black hole to our scenery. An example of the same picture with and without the black hole is shown on Figure 6. The observer is looking toward latitude -32.4° with respect to the coordinate grid we use. The visualization parameters are the same as previously. From a visual point of view, the central region of the picture, where we put the black hole, is now scattered all around (and at some distance) of the black hole silhouette. Therefore, the presence of the black hole somehow pushes away the background image as compared to the

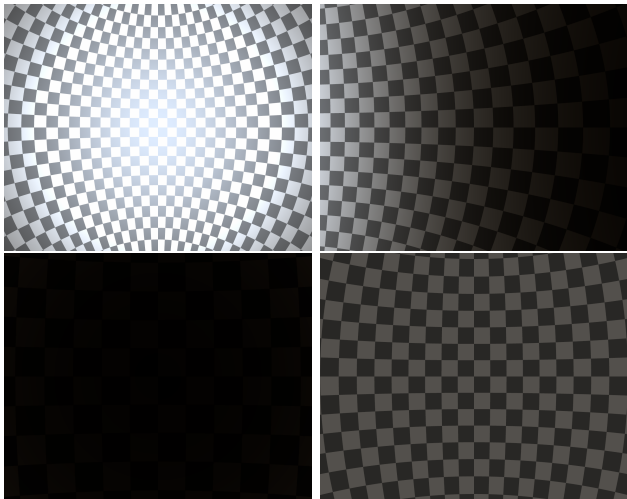


FIG. 5. Same as Fig. 3 and 4 above, but now including everything. The luminosity gradient as one goes from the front to rear direction is even larger than previously since, in addition to that of the previous figure, it is modulated by an extra $(1+z)^{-4}$ factor, which here, for $v = 0.5$, varies from 9 in the front direction to $1/9$ in the rear one.

undistorted case. Between the distorted image of the celestial sphere lies a series of ghost images which we will discuss below.

2. Gravitational blueshift

For any geodesic, the quantity $E \equiv \dot{t}(1 - 2M/r)$ is conserved. If we denote by ω the \dot{t}_γ component of some null geodesic, which means that the frequency of such photon measured by an observer at rest and at infinity will be ω , then the \dot{t}_γ component will be everywhere given by

$$\dot{t}_\gamma = \frac{\omega}{1 - \frac{2M}{r}}. \quad (70)$$

Now, if we consider a static lightlike observer outside the blackhole, the only component of his/her 4-velocity will be $\dot{t}_{\text{obs}} = (1 - 2M/r)^{-\frac{1}{2}}$. Therefore the frequency that such observer crossing a null geodesic coming from infinity will be

$$\omega' = g_{tt}\dot{t}_\gamma\dot{t}_{\text{obs}} = \omega\dot{t}_{\text{obs}} = \omega \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}. \quad (71)$$

The corresponding blueshift that will be measured is therefore

$$z_{\text{grav}} = -1 + \frac{\omega}{\omega'} = -1 + \sqrt{1 - \frac{2M}{r}}. \quad (72)$$

A few pictures on the increasing blueshift (and hence, the increasing brightness) as a static observer approaches to horizon is shown on Fig. 7

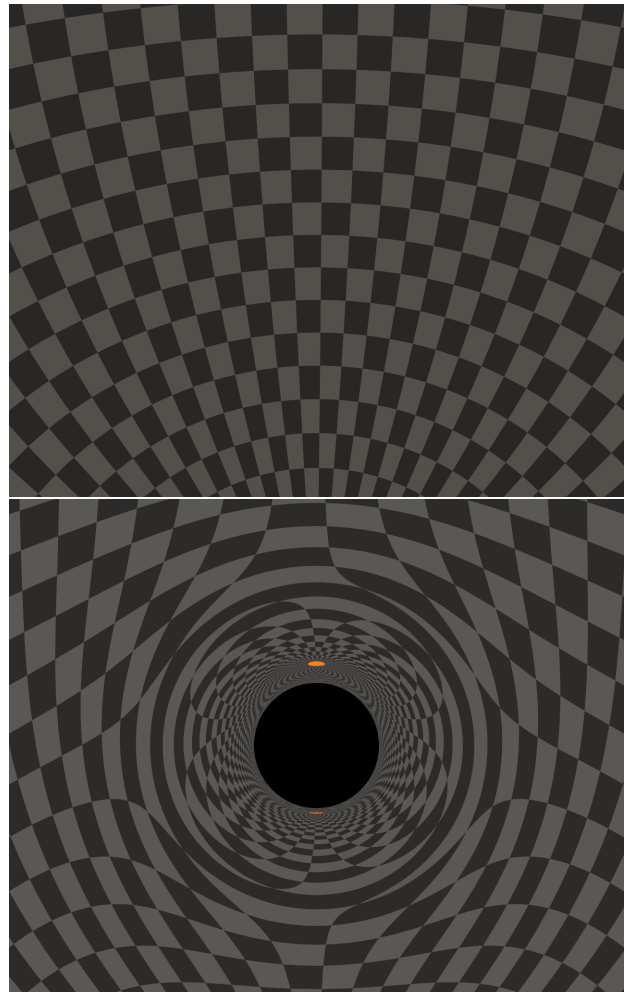


FIG. 6. Comparison of a view seen by a static observer with and without the presence of the black hole. In the latter case, the observer lies at coordinate $r = 30M$. The black hole lies in front of the most central dark square of the top picture. This square is seen highly distorted surrounding the silhouette of the black hole in the second picture. The two poles of the coordinate grid are seen as secondary/ghost images between the silhouette and the distorted black square, the south pole ghost image appearing above and that of the north pole below. Note that the second image is very slightly brighter than the first one because we have included the gravitational blueshift, which here takes the value $z = \sqrt{1 - 2M/r} - 1 \sim -0.0339$. However strange it may look like, such picture is nothing more than the visual translation of the deflection function shown in Fig. 1.

3. Lensing

As it is obvious from the distortion of central dark square of Fig. 6, the distortion of image induced by the black hole also modifies the angular size of background objects. This is the so-called gravitational lensing effect. In order to illustrate it, we reproduce the second image of Fig. 6 by modulating its intensity by the factor f de-

finned in Eq. (51). The result is shown in Fig. 8. As is well known, the point of the celestial sphere which happens to be exactly behind the black hole experiences an infinite amplification, at least as long as one considers geometric optics, and appears as an infinitely bright circle around the black hole, i.e., the so-called Einstein ring. The opposite point on the celestial sphere experiences the same behaviour and can be seen much closer to the black hole silhouette as shown in the accompanying Figure.

4. Multiple images

A star that lies close to the direction that is exactly behind the black hole from the observer point of view will show two distinct images, corresponding to light rays that pass on each side of the black hole, while belonging to the plane containing the black hole, the observer and the star. These sets of double images can be easily spotted on any image, especially when the star direction is sufficiently close to the observer-black hole axis, so that each star image experiences lensing. However, even when this is no longer the case, double images can be spotted, although with more difficulty. It is to be noted that because geodesics in the Schwarzschild metric are planar, each star image lies on a great circle on which also lies the null radial geodesic going from the observer to the black hole. If one uses stereographic projection, these three points lie along a straight line along the screen if seen by a static observer. Figure 9 shows an example of multiple image in a simulated, visually realistic astronomical background. In some rare occasion, one can see more than two images of a given star. If the observer does not stand too close to the black hole, this occurs mostly for stars that lie along the observer-black hole axis, as it is the case for star HD49359 of Fig. 9. Zoomed-in versions of this view showing two extra ghost images of this star are shown in Fig. 10. The amount of deflection a light ray experiences when approaching the black hole increases indefinitely as its impact parameters approach the critical value $3\sqrt{3}M$ (see Fig. 1). Consequently, there exists an infinite number of multiple images, all of which appear increasingly close to the edge of the black hole silhouette. Such images can be seen only if one zooms in by a fairly large factor, as exemplified in Fig. 11.

C. Combined effects

1. Circular orbits

One can combine special and general relativistic effects by considering an observer who is moving around a black hole. The simplest example corresponds to that of a circular orbit. As is well known, a timelike observer around a Schwarzschild black hole experiences a radial potential

of the form

$$V(r) \propto \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{r^2}\right), \quad (73)$$

where L is the observer angular momentum per unit of mass. A stable circular orbit corresponds to the local minima of V , which correspond of the largest of the roots of equation

$$Mr^2 - L^2r + 3L^3M = 0, \quad (74)$$

whose smallest value is $r = 6M$. For a fixed orbit radius, the observer's four-velocity u_{circ}^μ is then given by its constants of motion, with

$$L^2 = \frac{Mr}{1 - \frac{3M}{r}}, \quad (75)$$

and

$$E \equiv g_{tt}u_{\text{circ}}^t = \frac{1 - \frac{2M}{r}}{\sqrt{1 - \frac{3M}{r}}}, \quad (76)$$

so that one has, assuming that the orbital plane lies within the equatorial $\theta = \pi/2$ plane,

$$u_{\text{circ}}^t = \frac{1}{\sqrt{1 - \frac{3M}{r}}}, \quad (77)$$

$$u_{\text{circ}}^\varphi = \frac{Mr^{-\frac{3}{2}}}{\sqrt{1 - \frac{3M}{r}}}. \quad (78)$$

The orbital velocity v_{circ} with respect to an observer situated at the same radial coordinate is then given by $u_{\text{stat},\mu}u_{\text{circ}}^\mu = \gamma = (1 - v_{\text{circ}}^2)^{-\frac{1}{2}}$, where u_{stat}^μ is the static observer four-velocity, whose only non zero component is $u_{\text{circ}}^t = (1 - 2M/r)^{-\frac{1}{2}}$. The velocity is then

$$v_{\text{circ}} = \sqrt{\frac{\frac{M}{r}}{1 - \frac{2M}{r}}}. \quad (79)$$

Neglecting the denominator gives the well-known formula of the non relativistic third Kepler law. For orbital radii of $r = 30M$ and $r = 6M$, the orbital velocity is therefore $v_{\text{circ}} \simeq 0.189c$ and $v_{\text{circ}} = 0.5c$, respectively. In order to determine the maximum redshift and blueshift an observer experiences, one has to combine the gravitational redshift formula (72) with the kinetic one (66), the combined redshift z_{tot} being given by

$$1 + z_{\text{tot}} = (1 + z_{\text{grav}})(1 + z_{\text{kin}}). \quad (80)$$

For an observer at $r = 6M$, the maximum blueshift (in the front direction) is thus $z_{\text{max}} = \sqrt{2}/3 - 1 \simeq -0.529$, and the maximum redshift (in the rear direction) is $z_{\text{min}} = \sqrt{2} - 1 \simeq 0.414$. Fig. 12 shows a comparison between the side view seen by a $r = 30M$ observer looking toward the black hole, and the front view seen by a $r = 6M$ observer, both in circular orbit.

2. Non circular, non radial trajectories

Let us switch to geodesic non circular trajectories. Relativistic equivalent of Newtonian parabolic trajectories correspond to trajectories with zero velocity at infinity but non zero angular momentum L per unit of mass. Simple algebra then show that periastron radial coordinate is given by

$$\frac{r_{\text{per}}}{M} = \frac{1}{\frac{1}{4} - \sqrt{\frac{1}{16} - \frac{M^2}{L^2}}}, \quad (81)$$

and velocity at periastron is

$$v_{\text{per}} = \sqrt{\frac{2M}{r}}. \quad (82)$$

Obviously, such a relativistic equivalent of a parabolic trajectory has a periastron bounded by $r = 4M$, and the maximal velocity at periastron is then $v = c/\sqrt{2}$. Now, the trajectory in itself turns much more around the black hole than a parabola because of the extreme relativistic shift of periastron it experiences (the so-called zoom-and-whirl effect), and the azimuthal angle shift between far from the black hole and periastron is much bigger than $\pi/2$. Fig. 13 gives an example of an almost extremal pseudo-parabolic trajectory.

One can also have trajectories which are the relativistic equivalent of Newtonian hyperbolic trajectories. In this case, a convenient set of parameters describing the trajectory are the velocity at infinity and the impact parameter. Conversely, one can determine which is the minimum velocity at infinity, v_{∞} , that allows to reach a given value of the radial r_{per} coordinate at periastron. After some algebra, it appears that⁴ this is conveniently done if we parametrize the periastron radial coordinate by the quantity e , $1 < e < 3$, such that

$$r_{\text{per}} = \frac{2(3+e)}{1+e}M, \quad (83)$$

then the velocity at infinity must be greater than

$$v_{\infty} > \sqrt{\frac{e^2 - 1}{8}}. \quad (84)$$

Equivalently, this minimum velocity can be expressed as a function of r_{per}/M :

$$v_{\infty} > \sqrt{\frac{4 - \frac{r_{\text{per}}}{M}}{\frac{r_{\text{per}}}{M} - 2}}. \quad (85)$$

Figures 14 and 15 show two example of such near extremal pseudo-hyperbolic trajectories.

D. Horizons

As long as the observer lies outside the horizon, any calculation can be done in the standard Schwarzschild coordinates, although this is not necessarily what we do in practice. Such a coordinate choice is no longer possible when the observer is within the horizon since all the null geodesics he or she intersects have crossed the horizon and consequently have locally necessitated to use another coordinate system.

We present in Fig. 16 three views of a radial geodesic trajectory starting from infinity and plunging into the black hole. Several interesting features can be seen from those images:

- Firstly, the angular size of the black hole is rather small at horizon crossing. This comes from the fact that an observer who is about to cross the horizon has a very large relative velocity with respect to a static observer close to the horizon. This means that the view seen by the former experiences a very strong aberration phenomenon with respect to the view seen by the latter, thus drastically reducing the black hole angular size. In order to determine the angular size of the black hole, one needs to know which angle δ a geodesic with critical impact parameter $L/E = \pm 3\sqrt{3}M$ makes with respect to the radial direction. We therefore have to consider Eq. (42) together with the constraint $|L/E| = 3\sqrt{3}M$, with L and E given by Eqns (21,20). At horizon crossing, when $r = 2M$ one obtains

$$\cos \delta_{\text{hor}} = \frac{23}{31}, \quad (86)$$

leading to an angular diameter of ~ 82.1 degrees.

- Secondly, soon before hitting the singularity, the same calculation also gives the surprisingly simple result

$$\cos \delta_{\text{sing}} = 0, \quad (87)$$

leading to an angular size of 180 degrees. In other words, hitting the singularity happens when the black hole silhouette fills exactly half of the celestial sphere, just as what would happen in a Euclidean space when one lands on the surface of the spherical body.

- Thirdly, the celestial sphere is never very dark before horizon crossing. In order to see this, one has to consider the scalar product of $u_{\text{obs}}^{\mu} k_{\mu}$ between the radially freely falling observer and a radial null geodesic. Very simple algebra then shows that the maximum redshift that is seen by the observer at the moment he or she crosses the horizon is 1. It is only when the observer is deep within the horizon that the maximum redshift significantly increases.

⁴ See Ref. [1] for more details.

In fact, soon before hitting the singularity, only a very thin ring of direction show a blueshift rather than a redshift, since all what is behind the observer is very dark, being a highly redshifted flux of radiation, and everything that is in front of the observer is absolutely dark, being the black hole silhouette.

VIII. CONCLUSION

In this paper, we have outlined the main steps in order to produce a correct rendering of relativistic raytracing in the Schwarzschild metric. Although this problem is not new, we have obtained a very satisfactory (and, to our knowledge, new) way to simulate correctly the rendering of stars without any significant increase of the CPU time. This was made possible thanks to the spherically symmetric character of the Schwarzschild metric and in fact, such a trick can be easily implemented for any other spherically symmetric metric such as a Reissner-Nordstrom one, some result of which will be presented elsewhere. Unfortunately, this does not work in a simple way in the Kerr metric or any other non spherically symmetric metric such as the Papapetrou-Majumdar one. Some results regarding these metrics will also be presented elsewhere.

One major drawback of the techniques presented here is that they are still computationally intensive and do not allow real-time rendering of the metric since even at moderate resolutions, the CPU time is of several dozens of seconds. Real-time rendering thus necessitate to reduce CPU time by a factor greater than 10^3 , which is a rather ambitious goal, the attainability of which will be presented in a future work.

Appendix: On the structure of null geodesics in the Schwarzschild metric

We briefly recall here the different types of null geodesics in the Schwarzschild metric as well as the parameters that allow to distinguish them.

A null geodesics described by wavevector k^μ is characterized by the trivial equation

$$g_{\mu\nu}k^\mu k^\nu = 0. \quad (\text{A.1})$$

Using the constants of motion E and L^2 defined in Eqns (20,21), this can be rewritten

$$E^2 - \dot{r}^2 = \frac{L^2}{r^2} \left(1 - \frac{2M}{r} \right), \quad (\text{A.2})$$

where for simplicity \dot{r} denotes the r component of the wavevector.

Such an equation can formally be seen as describing the motion of a one dimensional particle along coordinate r , with a total energy $E^2/2$ and subject to a potential $V(r)$ given by

$$V(r) = \frac{1}{2} \frac{L^2}{r^2} \left(1 - \frac{2M}{r} \right). \quad (\text{A.3})$$

This potential goes to 0 at infinity and tends to minus infinity when r goes to 0. It has a unique maximum at $r = 3M$ (regardless of the value of L^2) whose value is $V_{\max} = L^2/54M^2$.

Consequently,

1. If $E^2/2 > V_{\max}$, or, equivalently $L/E < 3\sqrt{3}M$, then the geodesic does not have any turning points along the r coordinate and therefore its endpoints are $r = 0$ and $r = \infty$. It originates from infinity if and only if $\dot{r} < 0$.
2. If $E^2/2 < V_{\max}$ then the geodesics has its both endpoints either at $r = 0$ or at $r = \infty$. In the first case, it always lies within the $[0, r_{\max} < 3M]$ region and in the second case it always lies in the $[r_{\min} > 3M, \infty[$ region. The values of r_{\max} and r_{\min} are computed by finding the positive roots of the third degree polynomial equation $E^2/2 = V(r)$, the solution of which can be expressed in term of moderately simple elementary functions which do not matter here since if one has $E^2/2 < V_{\max}$ a geodesic originates from infinity if and only if one lies at $r > 3M$.
3. The critical case $L/E = 3\sqrt{3}M$ corresponds to geodesics which are stuck at $r = 3M$ or which indefinitely spiral toward this value, either originating from $r = 0$ or from $r = \infty$. In practice such geodesics never have to be taken care of since numerical roundoff error prevent from having this exact value of the L/E ratio even if one tries to.

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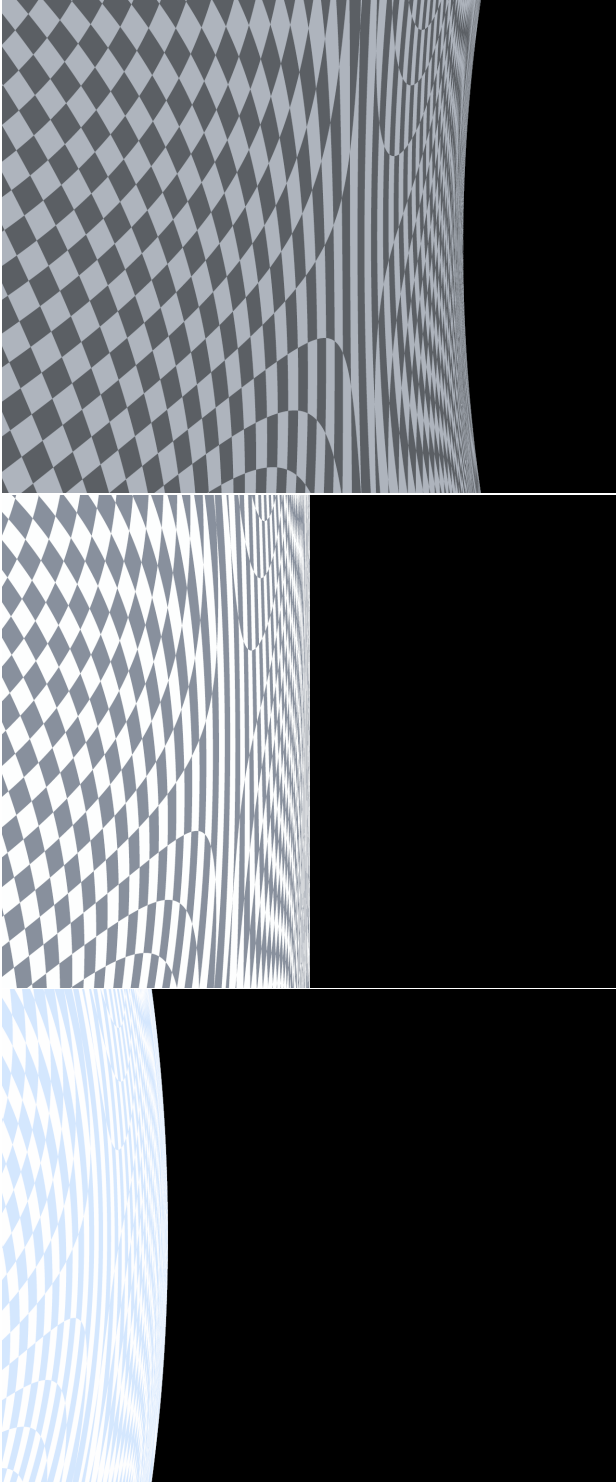


FIG. 7. Three views of the vicinity of the black hole as seen by a static observer that stands at $r = 4M$, $r = 3M$, and $r = 2.5M$ (from top to bottom). As is well-known, the $r = 3M$ radial coordinate corresponds to that where photons can have (unstable) circular orbits around the black hole, so that in practice the black hole silhouette spreads over a half sphere for a static observer. Below this value of r , the silhouette no longer looks convex, but concave instead. The increasing brightness as r decreases is due to the increasing blueshift z_{grav} , which takes values $\simeq -0.293$, $\simeq -0.423$ and $\simeq -0.553$, respectively.

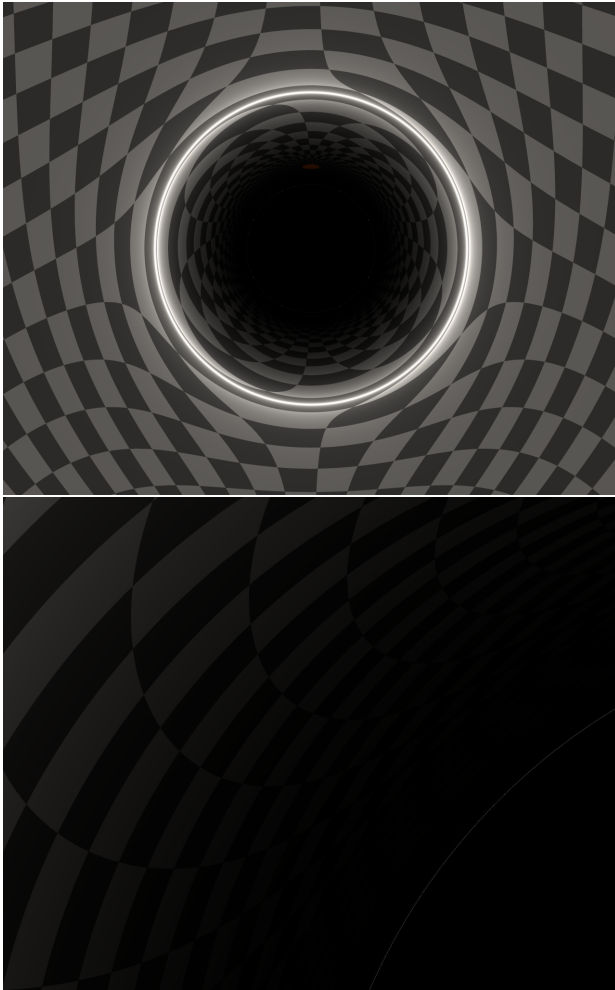


FIG. 8. An illustration of the amplification and de-amplification phenomena induced by deflection of light. Standard calculation indicate that the region opposite to the black hole is slightly de-amplified, whereas amplification increases as one approaches the Einstein ring. Within the Einstein ring, amplification decreases drastically and de-amplification kicks in till one goes very close to the black hole silhouette (top picture). Then, a tiny region of amplification appears, which corresponds to the Einstein ring of the black hole anticerter. Because the width on the second amplification region is extremely small, one needs a fairly large zooming factor in order to see it (bottom picture, showing the upper left quadrant of the black hole silhouette).

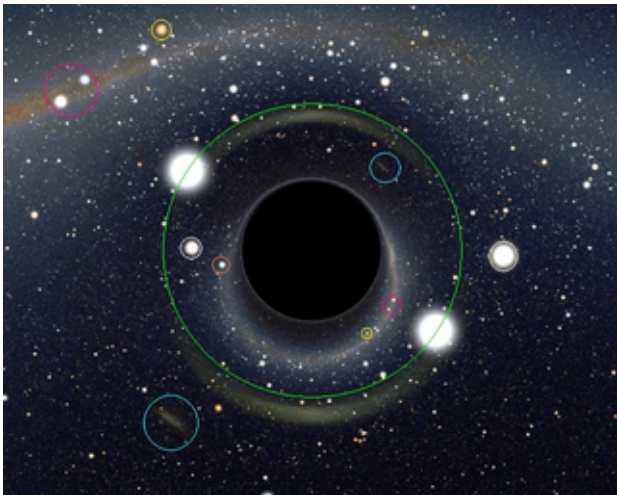


FIG. 9. A few examples of multiple images in an astronomically realistic simulated view. In the upper left quadrant, the circled pair of stars correspond to α and β Centauri, and the single circled star is γ Crucis. In the lower left quadrant, the circled structure is the Small Magellanic cloud, the Large Magellanic cloud being the very large U-shaped structure above and below the black hole silhouette. The bright, circled star to the right is α Carinae. Each pair of these stars are, for one outside the Einstein ring (large circle around the black hole silhouette), and for the other, inside. The only exception is for a star that is almost exactly behind the black hole, whose two images lie (almost) exactly on the Einstein ring. This is the case for the highly lensed, otherwise anonymous mag 7 star HD49359. The circled star inside the Einstein ring whose primary image is not seen is Sirius, whose primary image is offscreen (too much to the right), but whose secondary image is still easily visible despite deamplification thanks to the very low magnitude of the unlensed star.



FIG. 10. When zooming in between HD49359 primary or secondary image and the silhouette of the black hole, one sees two ghost images of this star, all of which lie within the segment joining the previous two images.

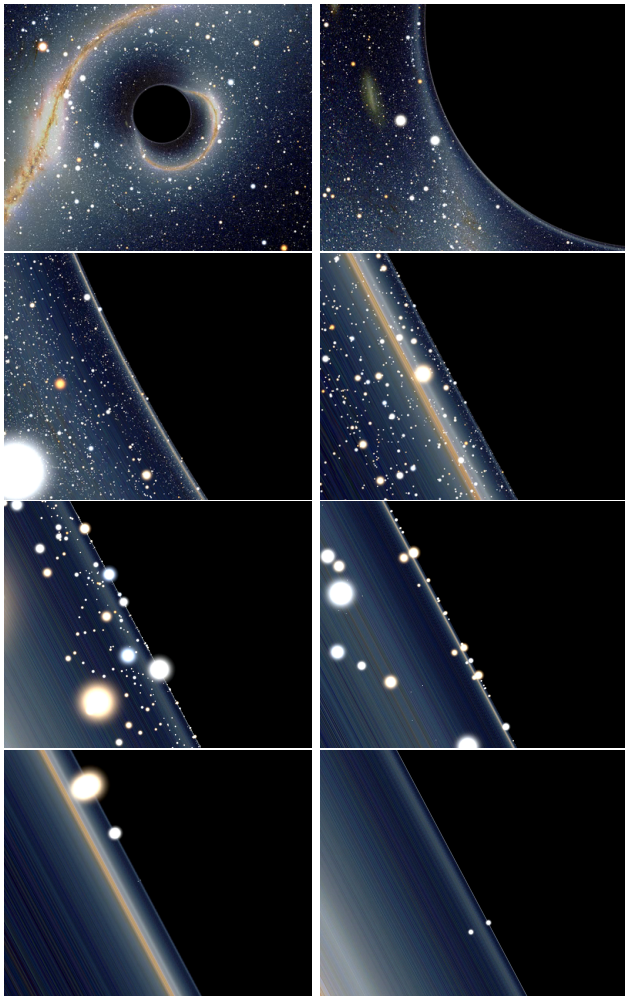


FIG. 11. An example of the multiple image phenomenon in the Schwarzschild metric. One starts with an image with a 90 degrees field of view, and each subsequent view (from left to right, then top to bottom) has a zoom factor of around 4 with respect to the previous one, so that the last one has a field of view around 160 mas. The first view shows the primary image of the Galactic disk, as well as its first “C”-shaped ghost image. The thin halo around the black hole corresponds to the second ghost image. The 2nd view shows part of the first ghost image of the disk, as well as the ghost image of the Large Magellanic cloud. The second ghost image is now seen to be slightly away from the edge of the black hole silhouette (especially in the bottom part of the view). In the 3rd view, one sees the first ghost image as a large band in the left of the image, and the second ghost image is now clearly visible. It is still visible in the 4th view, which also shows the third ghost image. Note that each successive ghost image is alternatively bright then dim because they correspond to opposite parts of the Galactic disk, whose luminosity is not uniform. In the 5th view, the third ghost image has been broadened on the left, whereas the fourth ghost image appears detached from the edge of the silhouette. This fourth image is now quite broadened on the 6th view, which show the fifth ghost image of the disk, close to the edge of the silhouette. This fifth ghost image is then seen on the 7th and 8th views, together with the sixth ghost image, and, in the last view, the seventh ghost image. Note that the number of stars decreases as one zooms in, because of the finiteness of our star catalog. A deeper star catalog would be obviously necessary for the last views. Also, assuming that stars are pointlike sources may be questionable here since at that resolution stars angular size might possibly become visible, and make them appear as very elongated thin segments.

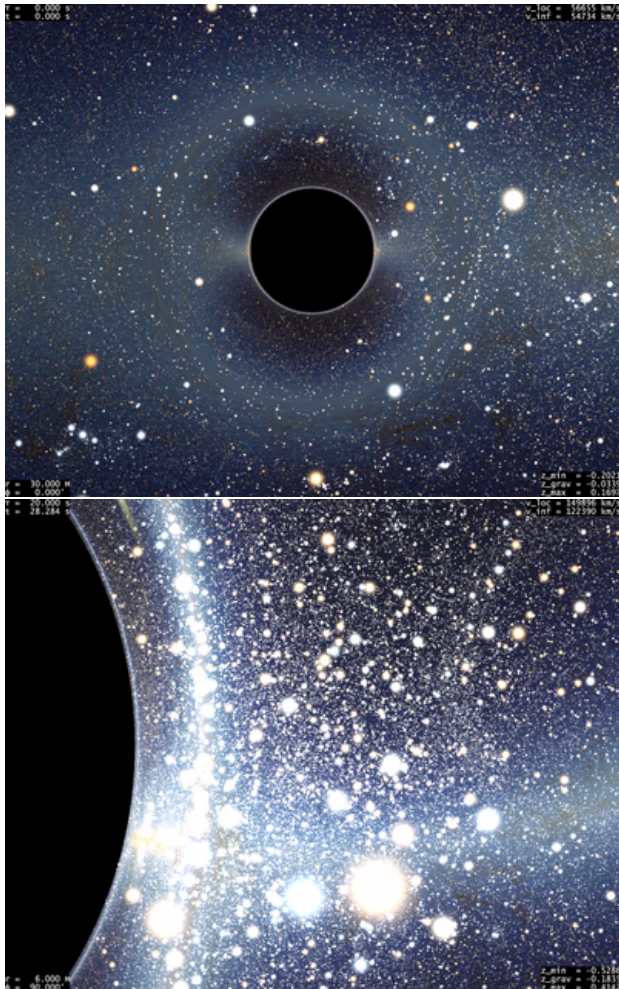


FIG. 12. Comparison of two circular orbits, with radii $r = 30M$ (top) and $r = 6M$ (bottom). In the first trajectory, the velocity is weakly relativistic, so that special relativistic effects are moderate. The direction of motion, which is on the right of the image, is only thus only moderately brighter than the opposite one. On the contrary, there is a very strong brightening in the second image, since the observer is now subject to a large kinetic blueshift (since orbital velocity is half of the speed of light here) and a significant gravitational blueshift (since the observer is close to the black hole). Similarly, the star background is fairly recognizable in the first picture. One can for example spot a somewhat flattened version of Orion constellation in the lower left quadrant of the image as well as a flattened Taurus un the bottom centre. This is no longer the case in the second one, where even familiar constellations are difficult to spot because the star background is now so crowded. Large blueshift allows some highly blueshifted cold stars to overcome the usually brighter stars we are used to see. There are, for example, many more bright, orange, stars (i.e., usually cold and faint stars) around Ursa Major constellation in the upper right quadrant of the image.

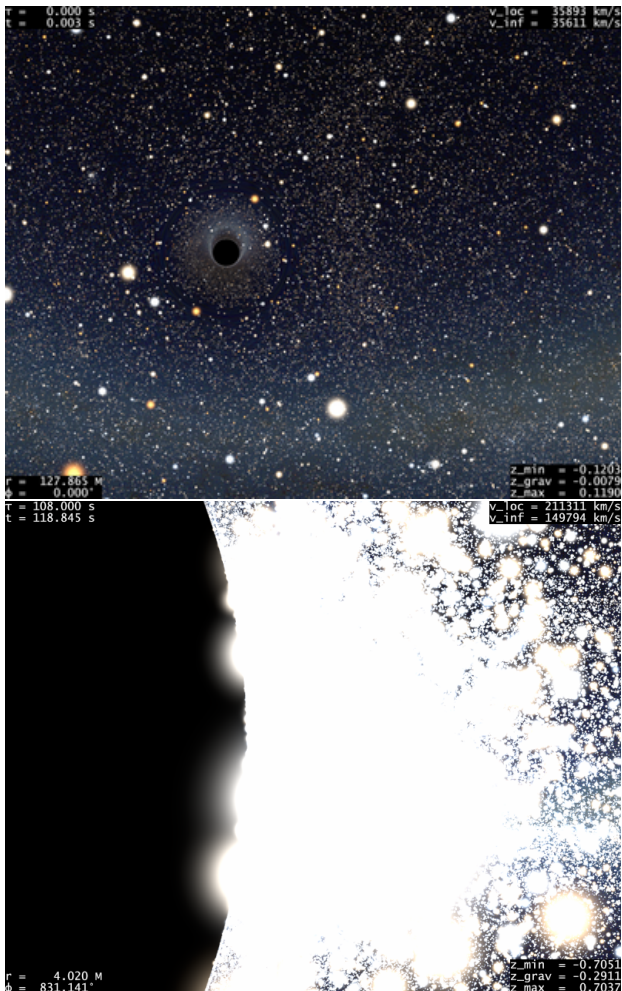


FIG. 13. Two views of an almost extremal pseudo parabolic orbit, starting from infinity a zero velocity and angular momentum close to the minimum allowed value of $4M$ (see Eq. (81)). Top image corresponds to a view far from the black hole, where the trajectory is close to radial. Bottom image corresponds to a view earlier than, but very close to periastron, at $r = 4.02M$. Local velocity, given by Eq. (82) is very close to $c/\sqrt{2}$, and velocity measured by a distant observer, given by $v_{inf} = v_{per}\sqrt{1 - 2M/r}$ is very close to $c/2$. Note that the azimuthal angle shift between the two view is ~ 831 deg., much larger than the Newtonian analog which could be slightly smaller than 180 degrees.

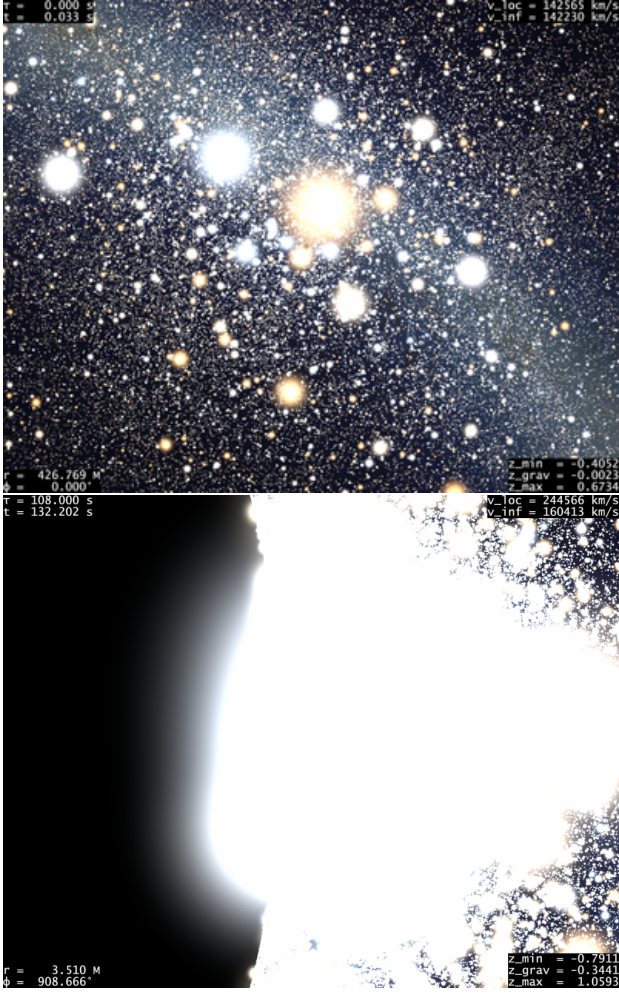


FIG. 14. Pseudo hyperbolic trajectory with a periastron radial coordinate of $3.5M$, which necessitates a velocity at infinity at least larger than $\sqrt{2}c/3 \sim 141323$ km/s (actual value chosen here is very slightly higher). A velocity close to that value is reached far from the black hole in the top view. Bottom view is computed close to periastron. The black hole mass is the same here as in the previous Figure, and the proper time interval between first and second view is also the same. Consequently, since bottom view in both figures are close to periastron, top view radial coordinate is large in this Figure than in the previous one because velocity far from the black hole is larger. Black hole angular size is in addition further reduced by the stronger aberration of this Figure, so that it is barely visible (below the brightest central, orange star). Note also that the background sky is much brighter, as expected.

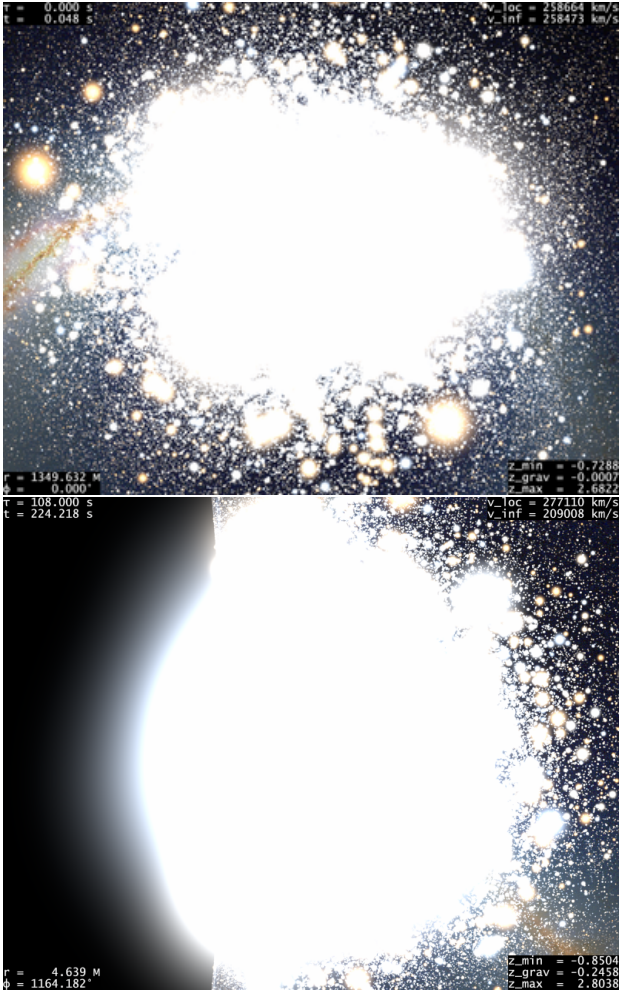


FIG. 15. Same as Fig. 14, except that periastron now occurs at $r = 3.1M$, which necessitates an initial velocity greater than $v \sim 0.862c \sim 258552$ km/s. On the top picture, the black hole angular size is so small and the star background so bright that the black hole silhouette is invisible with our rendering choices.

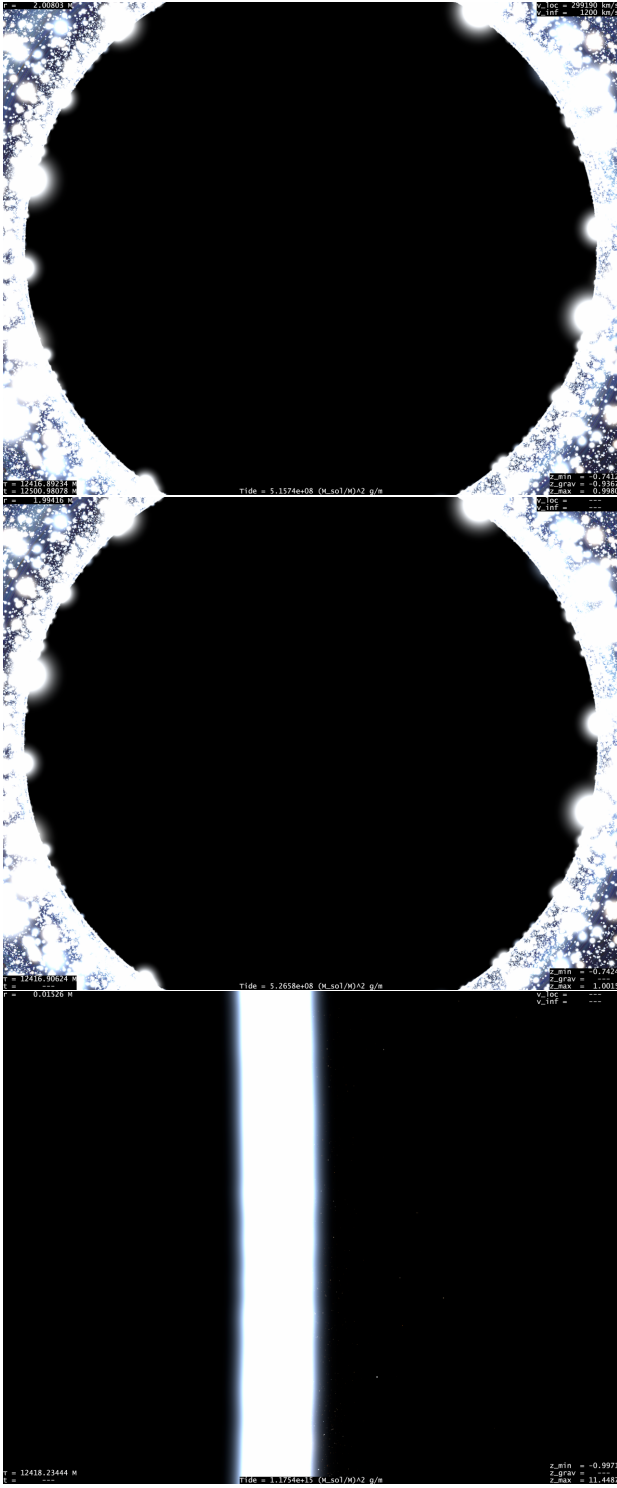


FIG. 16. Three views of a radial geodesic trajectory plunging into the black hole. Top and middle image show the front view soon before and soon after horizon crossing. As expected, no visual hint allows to decide easily whether or not the observer has crossed the horizon. Bottom image shows the side view soon before hitting the singularity.